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AN ANALYTICAL SOLUTION FOR THE SELF- AND MUTUAL RADIATION RESISTANCES OF A RECTANGULAR PLATE

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It is widely accepted that, for an arbitrary acoustic wavenumber, the radiation resistance of a simply supported rectangular plate has to be calculated through numerical integration. In this study, an analytical solution for the self- and mutual radiation resistances is obtained in the form of the power series of the non-dimensional acoustic wavenumber. Unlike the previous analytical or asymptotic solutions, it is not subject to any of the restrictions usually imposed upon the acoustic wavenumber. A few numerical examples are given simply to verify the solution and elucidate the reliance of the self- and mutual radiation resistances on the acoustic wavenumber and the plate aspect ratio. It is shown that the present formulae are extremely efficient by cutting the CPU time by orders of magnitude in comparison with the traditional numerical integration scheme. This investigation successfully fills the long-existing gap in solutions for the moderate wavenumbers that are often of primary concern in an acoustic analysis.

1. INTRODUCTION

Acoustic radiation from vibrating plates has been an active research subject for many years. As early as 1960s, Maidanik [1] proposed several approximate formulae for calculating the radiation resistance in different wavenumber regions known as well-below, about and above coincidence. Wallace [2] later introduced the next higher order term into the solution for the well-below coincidence case and investigated the effects on the radiation resistance of the inter-nodal areas and their aspect ratios. Davies [3] calculated the radiation resistance for the acoustically slow modes by allowing acoustic wavelength to be comparable to the plate dimensions. Williams [4] developed a couple of series expansions (in ascending powers of the acoustic wavenumber) for the acoustic power radiated from a planar source. Specifically, the series expansion in terms of the Fourier transformed velocity and its derivatives in wavenumber space was used to derive approximate expressions for the power radiated at low frequencies from a mode of a rectangular plate with three different types of boundary conditions (simply supported, clamped–clamped, and free–free). Levine [5] and Leppington *et al.* [6] derived several asymptotic formulae for the large acoustic wavenumbers. Recently, Li and Gibeling [7] gave an asymptotic solution for the modes of a large modal index by making use of the principle of the stationary phase.

In the literature, most investigations have been focused on the self-radiation resistance (or the so-called modal radiation efficiency/coefficient) of an individual mode. Accordingly, the total power radiated from a plate is normally calculated by simply adding up the powers

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independently produced by each mode. The risk of neglecting the cross-modal couplings has actually long been realized and the reason for doing so is perhaps simply due to the common belief that it may otherwise become a tremendous computational burden. Based on William's results, Snyder and Tanaka [8] derived a set of simple formulae for calculating the mutual radiation resistances directly from the self terms for small wavenumbers. Li and Gibeling [9] developed an algorithm for an easy and accurate determination of the mutual radiation resistances in the whole frequency range and demonstrated that the cross-modal couplings could have a meaningful impact on the radiated sound power, even at a resonant frequency. Consideration of the cross-modal couplings is also of significance to a class of problems involving plates loaded with features like masses or springs. It has been shown in reference [6] that the acoustic characteristics of a spring-reinforced plate can be readily determined from the self- and mutual radiation resistances of the corresponding simple/unloaded plate. Certainly, the effects of the cross-modal couplings are of direct concerns in fluid–structure interactions [10–13].

Mathematically, the radiation resistance of a baffled rectangular plate can be expressed as either a double integral over a hemisphere of a sufficiently large radius in a farfield formulation or a quadruple integral over the plate surface in a nearfield formulation. These integrals may typically render some kinds of asymptotic or approximate estimates for the extreme (i.e., sufficiently large and small) acoustic wavenumbers. However, it has been widely believed that the radiation resistance can only be determined numerically for the moderate acoustic wavenumbers. In this paper, based on the MacLaurin expansion of the Green's function previously used in reference [4], a new simple analytical solution is derived for the self- and mutual radiation resistances of a rectangular plate. The issues related to its numerical calculation are then discussed in detail. Finally, numerical examples are given primarily to verify the accuracy and efficiency of the current solution.

2. THE SELF- AND MUTUAL RADIATION RESISTANCES OF A SIMPLY SUPPORTED RECTANGULAR PLATE

2.1. BASIC EQUATIONS

Consider a rectangular plate simply supported in an infinite rigid baffle, as shown in Figure 1. The total acoustic power radiated from the plate can be calculated from

$$W = \frac{1}{2}\int_S \Re\,[\dot{w}^*(\mathbf{x}),\, p(\mathbf{x})]\,\mathrm{d}\mathbf{x}, \tag{1}$$

where $p(\mathbf{x})$ is the sound pressure on the surface of the plate, $\dot{w}(\mathbf{x})$ is the normal velocity of the plate, and $\Re$ and $*$ denote the real part and the complex conjugate of a complex number respectively.

The normal or flexural displacement is sought here as a superposition of the plate modes

$$w(\mathbf{x}) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\,\psi_{mn}(\mathbf{x}) \tag{2}$$

or, in a matrix form,

$$w = \mathbf{\Psi}^{\mathrm{T}}\mathbf{A}, \tag{3}$$

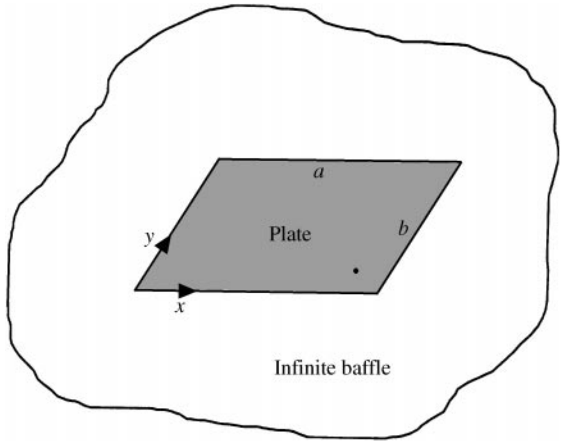

Figure 1. A baffled rectangular plate.

where A_{mn} is the modal co-ordinate and

$$\psi_{mn}(\mathbf{x}) = \frac{2}{\sqrt{ab}} \sin\frac{m\pi}{a} x \sin\frac{n\pi}{b} y. \tag{4}$$

The sound pressure on the plate surface can be determined from Rayleigh integral,

$$p(\mathbf{x}) = \frac{\mathrm{i}\omega\rho_0}{2\pi}\int_S \frac{\dot{w}(\mathbf{x}')\mathrm{e}^{-\mathrm{i}kr}}{r}\,\mathrm{d}\mathbf{x}', \tag{5}$$

where $k = \omega/c$ is the acoustic wavenumber, ω is frequency in radian, c is the speed of sound, ρ_0 is the density of air, and $r = |\mathbf{x} - \mathbf{x}'| = \sqrt{(x - x'^2 + (y - y')^2}$.

Substituting equations (3) and (5) into equation (1) results in

$$W = \tfrac{1}{2}\rho_0 c\omega^2 \mathbf{A}^{\mathrm{H}}\boldsymbol{\Xi}\mathbf{A}, \tag{6}$$

where H denotes the Hermitian of a matrix and $\boldsymbol{\Xi}$ is the specific radiation resistance matrix whose elements are defined as

$$\xi_{mn,m'n'} = \frac{2k}{\pi ab}\int_0^b\int_0^a\int_0^b\int_0^a \sin\alpha_m x \sin\beta_n y \sin\alpha_{m'}x' \sin\beta_{n'}y' \times \frac{\sin k\sqrt{(x-x')^2+(y-y')^2}}{\sqrt{(x-x')^2+(y-y')^2}}\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}x\,\mathrm{d}y, \tag{7}$$

where $\alpha_m = m\pi/a$ and $\beta_n = n\pi/b$.

In the literature, $\xi_{mn,m'n'}$ is usually referred to as the (self) radiation resistance if $m' = m$ and $n' = n$, and the mutual radiation resistance otherwise. The self-radiation resistance measures the effectiveness of an individual mode in generating sound, and the mutual radiation resistance determines how the sound field produced by one mode can affect the vibration of another mode.

It is clear from equation (6) that the self- and mutual radiation resistances both generally contribute to the total sound power. The major diagonal elements of the radiation resistance matrix Ξ represent the self-radiation resistances and all the others are the mutual radiation resistances. It is well known that the cross-modal coupling only occurs between a pair of modes which have the same parity indices in both the x and y directions. Hence, only about a quarter of the off-diagonal elements is not constantly zero. Further, it is clear from equation (7) that

$$\xi_{mn,m'n'} = \begin{cases} \xi_{m'n',mn}, \\ \xi_{mn',m'n}, \\ \xi_{m'n,mn'}. \end{cases} \tag{8}$$

Despite these favorable facts about the resistance matrix, the calculation of the mutual radiation resistance is still often considered an overwhelming task. As a result, in most investigations, the contributions of the mutual radiation resistances are simply ignored, assuming that they are not as important as the self terms. Since this assumption is not readily verified *a priori*, it is quite risky to routinely neglect the effects of the cross-modal coupling.

The integral in equation (7) is usually calculated by using some numerical integration techniques. In order to lessen the computational burden, a co-ordinate transformation technique can be used to recast the quadruple integral into several double integrals [5, 6]. Accordingly, the self- and mutual radiation resistances can be expressed as [7]

(1) for $m = m'$ and $n = n'$,

$$\xi_{mn,mn} = \frac{2k}{\pi ab}\left\{\frac{1}{\alpha_m\beta_n} J_1^{mn} + J_2^{mn} + \frac{1}{\alpha_m} J_3^{mn} + \frac{1}{\beta_n} J_4^{mn}\right\}, \tag{9}$$

(2) for $m \neq m'$ and $n \neq n'$,

$$\xi_{mn,m'n'} = \frac{2k}{\pi ab}\frac{\varepsilon(m'-m)\varepsilon(n'-n)}{(\alpha_m^2-\alpha_{m'}^2)(\beta_n^2-\beta_{n'}^2)}\{\alpha_m\beta_n J_1^{m'n'} - \alpha_m\beta_{n'} J_1^{m'n} - \alpha_{m'}\beta_n J_1^{mn'} + \alpha_{m'}\beta_{n'} J_1^{mn}\}, \tag{10}$$

(3) for $m \neq m'$ and $n = n'$,

$$\xi_{mn,m'n} = \frac{2k}{\pi ab}\frac{\varepsilon(m'-m)}{(\alpha_m^2-\alpha_{m'}^2)}\left\{\alpha_m J_3^{m'n} - \alpha_{m'} J_3^{mn} + \frac{\alpha_m}{\beta_n} J_1^{m'n} - \frac{\alpha_{m'}}{\beta_n} J_1^{mn}\right\}, \tag{11}$$

(4) for $m = m'$ and $n \neq n'$,

$$\xi_{mn,mn'} = \frac{2k}{\pi ab}\frac{\varepsilon(n'-n)}{(\beta_n^2-\beta_{n'}^2)}\left\{\beta_n J_4^{mn'} - \beta_{n'} J_4^{mn} + \frac{\beta_n}{\alpha_m} J_1^{mn'} - \frac{\beta_{n'}}{\alpha_m} J_1^{mn}\right\}, \tag{12}$$

where

$$\begin{Bmatrix} J_1^{mn} \\ J_2^{mn} \\ J_3^{mn} \\ J_4^{mn} \end{Bmatrix} = \int_0^b\int_0^a \begin{Bmatrix} 1 \\ (a-x)(b-y) \\ (b-y) \\ (a-x) \end{Bmatrix} \times \begin{Bmatrix} \sin\alpha_m x \sin\beta_n y \\ \cos\alpha_m x \cos\beta_n y \\ \sin\alpha_m x \cos\beta_n y \\ \cos\alpha_m x \sin\beta_n y \end{Bmatrix} \frac{\sin k\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\,\mathrm{d}x\,\mathrm{d}y \tag{13}$$

and

$$\varepsilon(m'-m)=\begin{cases}1 & \text{for } m'=m,\\ 0 & \text{for } m'-m=\pm 1,\ \pm 3,\ \pm 5,\dots,\\ 2 & \text{for } m'-m=\pm 2,\ \pm 4,\ \pm 6,\dots.\end{cases} \tag{14}$$

2.2. DERIVATION OF AN ANALYTICAL SOLUTION

An analytical solution in the form of the power series of the non-dimensional acoustic wavenumber ka will be derived below for the self- and mutual radiation resistances.

Making use of the series expansions previously used by Williams [4]:

$$\frac{\sin k\rho}{\rho}=\sum_{p=0}^{\infty}\frac{(-1)^p(k\rho)^{2p+1}}{\rho(2p+1)!}=\sum_{p=0}^{\infty}\frac{(-1)^p k^{2p+1}\rho^{2p}}{(2p+1)!}\quad (\rho=\sqrt{x^2+y^2}) \tag{15}$$

and

$$\rho^{2p}=(x^2+y^2)^p=\sum_{q=0}^{p}\binom{p}{q}x^{2p-2q}y^{2q}, \tag{16}$$

where

$$\binom{p}{q}=\frac{p!}{q!(p-q)!},$$

one will be able to obtain that

$$\begin{aligned}
J_1^{mn}&=\int_0^b\int_0^a \sin\alpha_m x\sin\beta_n y\,\frac{\sin k\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\,\mathrm{d}x\,\mathrm{d}y\\
&=\sum_{p=0}^{\infty}\sum_{q=0}^{p}\binom{p}{q}\frac{(-1)^p k^{2p+1}}{(2p+1)!}\int_0^b\int_0^a \sin\alpha_m x\sin\beta_n y x^{2p-2q}y^{2q}\,\mathrm{d}x\,\mathrm{d}y\\
&=\sum_{p=0}^{\infty}\sum_{q=0}^{p}\binom{p}{q}\frac{(-1)^p k^{2p+1}}{(2p+1)!}\left\{\int_0^a x^{2p-2q}\sin\alpha_m x\,\mathrm{d}x\int_0^b y^{2q}\sin\beta_n y\,\mathrm{d}y\right\}\\
&=\sum_{p=0}^{\infty}\sum_{q=0}^{p}\binom{p}{q}\frac{(-1)^p (ak)^{2p+1}b}{(2p+1)!}\,\sigma^{2q}S_{2p-2q}^{m}S_{2q}^{n},
\end{aligned} \tag{17}$$

where $\sigma=b/a$ is the plate aspect ratio, and

$$S_p^m=\int_0^1 x^p\sin m\pi x\,\mathrm{d}x. \tag{18}$$

It is easy to verify that

$$S_0^m=\int_0^1\sin m\pi x\,\mathrm{d}x=\frac{1+(-1)^{m+1}}{m\pi},\qquad S_1^m=\int_0^1 x\sin m\pi x\,\mathrm{d}x=\frac{(-1)^{m+1}}{m\pi} \tag{19, 20}$$

and the recursive relationship

$$S_p^m = \int_0^1 \sin m\pi x\, x^p\, dx = -\frac{x^p \cos m\pi x}{m\pi}\bigg|_{x=1} + \frac{p}{m\pi}\int_0^1 \cos m\pi x\, x^{p-1}\, dx$$

$$= \frac{(-1)^{m+1}}{m\pi} + \frac{p x^{p-1}\sin m\pi x}{(m\pi)^2}\bigg|_{x=1} - \frac{p(p-1)}{(m\pi)^2}\int_0^1 \sin \alpha_m x x^{p-2}\, dx$$

$$= \frac{(-1)^{m+1}}{m\pi} - \frac{p(p-1)}{m^2\pi^2} S_{p-2}^m \quad \text{for } p \geqslant 2. \tag{21}$$

Similarly, the second integral in equation (13) can be expressed as

$$J_2^{mn} = \int_0^b \int_0^a (a-x)(b-y)\cos\alpha_m x \cos\beta_n y \frac{\sin k\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\, dx\, dy$$

$$= \sum_{p=0}^{\infty}\sum_{q=0}^{p}\binom{p}{q}\frac{(-1)^p k^{2p+1}}{(2p+1)!}\int_0^b\int_0^a (a-x)(b-y)\cos\alpha_m x\cos\beta_n y\, x^{2p-2q} y^{2q}\, dx\, dy$$

$$= \sum_{p=0}^{\infty}\sum_{q=0}^{p}\binom{p}{q}\frac{(-1)^p k^{2p+1}}{(2p+1)!}\left\{\int_0^a x^{2p-2q}(a-x)\cos\alpha_m x\, dx \int_0^b y^{2q}(b-y)\cos\beta_n y\, dy\right\}$$

$$= \sum_{p=0}^{\infty}\sum_{q=0}^{p}\binom{p}{q}\frac{(-1)^p (ak)^{2p+1} ab^2}{mn\pi^2(2p+1)!}\ \sigma^{2q}(R_{2p-2q}^m - R_{2p-2q+1}^m)(R_{2q}^n - R_{2q+1}^n), \tag{22}$$

where

$$R_p^m = (m\pi)\int_0^1 x^p \cos m\pi x\, dx. \tag{23}$$

Also, it is readily verified that

$$R_0^m = \int_0^1 \cos m\pi x\, dx \equiv 0, \tag{24}$$

$$R_1^m = (m\pi)\int_0^1 x\cos m\pi x\, dx = -\frac{1+(-1)^{m+1}}{m\pi} \tag{25}$$

and, for $p \geqslant 2$,

$$R_p^m = \frac{p(-1)^m}{(m\pi)} - \frac{p(p-1)}{(m\pi)^2} R_{p-2}^m. \tag{26}$$

The remaining two integrals in equation (13) can be dealt with in the same manner. From equations (19–21) and (24–26) one will have

$$R_{2p}^m = \sum_{r=1}^{p} \frac{(-1)^{m+r+1}(2p)!}{(2p-2r+1)!\,(m\pi)^{2r-1}}, \tag{27}$$

$$R_{2p-1}^m = \sum_{r=1}^{p}\left\{\frac{(-1)^{m+r+1}(2p-1)!}{(2p-2r)!\,(m\pi)^{2r-1}}\right\} + (-1)^p\frac{(2p-1)!}{(m\pi)^{2p-1}}, \tag{28}$$

$$S_{2p-1}^{m} = \sum_{r=1}^{p} \frac{(-1)^{m+r}(2p-1)!}{(2p-2r+1)!\,(m\pi)^{2r-1}} \tag{29}$$

and

$$S_{2p}^{m} = \sum_{r=1}^{p} \left\{ \frac{(-1)^{m+r}(2p)!}{(2p-2r+2)!\,(m\pi)^{2r-1}} \right\} + (-1)^{p}[1-(-1)^{m}]\,\frac{(2p)!}{(m\pi)^{2p+1}}. \tag{30}$$

Now, substitution of equations (17) and (22) (and the similar ones corresponding to the last two integrals in equation (13)) into equations (9–12) will lead to

$$\begin{aligned}\xi_{mn,mn} = {} & \frac{2}{mn\pi^3} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \binom{p}{q} \frac{(-1)^p (ak)^{2p+2}}{(2p+1)!} \sigma^{2q+1} \\ & \times \{ S_{2p-2q}^{m} S_{2q}^{n} + (R_{2p-2q}^{m} - R_{2p-2q+1}^{m})(R_{2q}^{n} - R_{2q+1}^{n}) \\ & + S_{2p-2q}^{m}(R_{2q}^{n} - R_{2q+1}^{n}) + (R_{2p-2q}^{m} - R_{2p-2q+1}^{m}) S_{2q}^{n} \},\end{aligned} \tag{31}$$

$$\begin{aligned}\xi_{mn,m'n'} = {} & \frac{2\varepsilon(m-m')\varepsilon(n-n')}{mn\pi^3\,[1-(m'/m)^2][1-(n'/n)^2]} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \binom{p}{q} \frac{(-1)^p (ak)^{2p+2}}{(2p+1)!} \sigma^{2q+1} \\ & \times \{ S_{2p-2q}^{m'} S_{2q}^{n'} - (n'/n) S_{2p-2q}^{m'} S_{2q}^{n} \\ & - (m'/m) S_{2p-2q}^{m} S_{2q}^{n'} + (n'm'/mn) S_{2p-2q}^{m} S_{2q}^{n} \},\end{aligned} \tag{32}$$

$$\begin{aligned}\xi_{mn,m'n} = {} & \frac{2\varepsilon(m-m')}{mn\pi^3\,[1-(m'/m)^2]} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \binom{p}{q} \frac{(-1)^p (ak)^{2p+2}}{(2p+1)!} \sigma^{2q+1} \\ & \times \{ S_{2p-2q}^{m'}(R_{2q}^{n} - R_{2q+1}^{n}) - (m'/m) S_{2p-2q}^{m}(R_{2q}^{n} - R_{2q+1}^{n}) \\ & - (m'/m) S_{2p-2q}^{m} S_{2q}^{n} + S_{2p-2q}^{m'} S_{2q}^{n} \}\end{aligned} \tag{33}$$

and

$$\begin{aligned}\xi_{mn,mn'} = {} & \frac{2\varepsilon(n-n')}{mn\pi^3\,[1-(n'/n)^2]} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \binom{p}{q} \frac{(-1)^p (ak)^{2p+2}}{(2p+1)!} \sigma^{2q+1} \\ & \times \{ (R_{2p-2q}^{m} - R_{2p-2q+1}^{m}) S_{2q}^{n'} - (n'/n)(R_{2p-2q}^{m} - R_{2p-2q+1}^{m}) S_{2q}^{n} \\ & + S_{2p-2q}^{m} S_{2q}^{n'} - (n'/n) S_{2p-2q}^{m} S_{2q}^{n} \}.\end{aligned} \tag{34}$$

Equations (31–34) can be further simplified to

$$\xi_{mn,mn} = \frac{2}{\pi} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \binom{p}{q} \frac{(-1)^p (ak)^{2p+2}}{(2p+1)!} \sigma^{2q+1}\; T_{2p-2q}^{m}\, T_{2q}^{n}, \tag{35}$$

$$\xi_{mn,m'n'} = \frac{2}{\pi} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \binom{p}{q} \frac{(-1)^p (ak)^{2p+2}}{(2p+1)!} \sigma^{2q+1}\; U_{2p-2q}^{mm'}\, U_{2q}^{nn'}, \tag{36}$$

$$\xi_{mn,m'n} = \frac{2}{\pi} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \binom{p}{q} \frac{(-1)^p (ak)^{2p+2}}{(2p+1)!} \sigma^{2q+1}\; U_{2p-2q}^{mm'}\, T_{2q}^{n} \tag{37}$$

and

$$\xi_{mn,mn'} = \frac{2}{\pi} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \binom{p}{q} \frac{(-1)^p (ak)^{2p+2}}{(2p+1)!} \sigma^{2q+1} \, T_{2p-2q}^{m} \, U_{2q}^{nn'} \tag{38}$$

by defining

$$U_{2p}^{mm'} = \frac{\varepsilon(m-m')}{\pi(m^2 - m'^2)} (mS_{2p}^{m'} - m' S_{2p}^{m}) \tag{39}$$

and

$$T_{2p}^{m} = (S_{2p}^{m} + R_{2p}^{m} - R_{2p+1}^{m})/(m\pi). \tag{40}$$

It is clear from equation (39) that

$$U_{2p}^{mm'} = U_{2p}^{m'm}. \tag{41}$$

Making use of equations (27–30), equation (40) can be rewritten as

$$T_{2p}^{m} = \sum_{r=1}^{p} \left\{ \frac{2r(2p)!(-1)^{m+r}}{(2p-2r+2)!\,(m\pi)^{2r}} \right\} + (-1)^p \left[1 - (-1)^m\right] \frac{(2p+2)(2p)!}{(m\pi)^{2p+2}}. \tag{42}$$

Equations (35–38) represent a set of analytical expressions for the self- and mutual radiation resistances of a rectangular plate. Because no approximation is involved in the above derivations, the current solution is good for any acoustic wavenumber.

2.3. AN ASYMPTOTIC FORM OF THE SOLUTION FOR SMALL WAVENUMBERS

For the purpose of cross check, let us now consider a limiting case of equations (35–38) for the small acoustic wavenumbers or low frequencies. By explicitly spelling out the first two terms in each of the equations, one will readily have

(1) if m and n are both odd integers,

$$\xi_{mn,mn} = \frac{32k^2ab}{m^2 n^2 \pi^5} \left\{ 1 - \frac{k^2 ab}{12} \left[\left(1 - \frac{8}{m^2\pi^2}\right)\frac{a}{b} + \left(1 - \frac{8}{n^2\pi^2}\right)\frac{b}{a} \right] \right\} + O(k^6) \tag{43}$$

and

$$\xi_{mn,m'n'} = \frac{32k^2ab}{mnm'n'\pi^5} \left\{ 1 - \frac{k^2 ab}{24} \left[\left(1 - \frac{8}{m^2\pi^2}\right)\frac{a}{b} + \left(1 - \frac{8}{m'^2\pi^2}\right)\frac{a}{b} + \left(1 - \frac{8}{n^2\pi^2}\right)\frac{b}{a} + \left(1 - \frac{8}{n'^2\pi^2}\right)\frac{b}{a} \right] \right\} + O(k^6) \tag{44}$$

(2) if m is odd and n is even,

$$\xi_{mn,mn} = \frac{8k^4ab^3}{3m^2 n^2 \pi^5} \left\{ 1 - \frac{k^2 ab}{20} \left[\left(1 - \frac{8}{m^2\pi^2}\right)\frac{a}{b} + \left(1 - \frac{24}{n^2\pi^2}\right)\frac{b}{a} \right] \right\} + O(k^8) \tag{45}$$

and

$$\xi_{mn,m'n'} = \frac{8k^4ab^3}{3mnm'n'\pi^5}\left\{1 - \frac{k^2ab}{40}\left[\left(1 - \frac{8}{m^2\pi^2}\right)\frac{a}{b} + \left(1 - \frac{8}{m'^2\pi^2}\right)\frac{a}{b} + \left(1 - \frac{24}{n^2\pi^2}\right)\frac{b}{a} + \left(1 - \frac{24}{n'^2\pi^2}\right)\frac{b}{a}\right]\right\} + O(k^8) \tag{46}$$

(3) if m and n are both even integers,

$$\xi_{mn,mn} = \frac{2k^6a^3b^3}{15m^2n^2\pi^5}\left\{1 - \frac{k^2ab}{28}\left[\left(1 - \frac{24}{m^2\pi^2}\right)\frac{a}{b} + \left(1 - \frac{24}{n^2\pi^2}\right)\frac{b}{a}\right]\right\} + O(k^{10}) \tag{47}$$

and

$$\xi_{mn,m'n'} = \frac{2k^6a^3b^3}{15mnm'n'\pi^5}\left\{1 - \frac{k^2ab}{56}\left[\left(1 - \frac{24}{m^2\pi^2}\right)\frac{a}{b} + \left(1 - \frac{24}{m'^2\pi^2}\right)\frac{a}{b} + \left(1 - \frac{24}{n^2\pi^2}\right)\frac{b}{a} + \left(1 - \frac{24}{n'^2\pi^2}\right)\frac{b}{a}\right]\right\} + O(k^{10}). \tag{48}$$

These equations are exactly the same as what were previously obtained by Wallace [1] and Snyder and Tanaka [13], respectively, for the self- and mutual radiation resistances, except for a minor difference in the coefficients for the second terms in equations (47) and (48). It has been verified numerically that the current expressions, equations (47) and (48), are more accurate.

Only the two lowest terms have been taken into account in each of the above equations. However, the next higher order term(s), if needed, can be readily included by making use of equations (35–38) together with equations (30), (39) and (42). Thus, given a frequency bound, equations (35–38) provide an easy and general way for deriving simple asymptotic or approximate formulae for computing the self- and mutual radiation resistances. Obviously, when more terms are included, the corresponding equations will accordingly allow a higher upper frequency limit which, as shown later, can be easily estimated from equation (54).

2.4. CONVERGENCE OF THE SOLUTION

For the large acoustic wavenumbers, many terms have to be used in equation (15). When p is a large number, the formulae used to calculate S^m_{2p} and T^m_{2p}, equations (30) and (42), tend to become numerically unstable due to round-off errors. This problem, however, can be easily overcome by using an alternative set of formulae:

$$S^m_{2p} = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}(m\pi)^{2r-1}}{(2r-1)!(2p+2r)} \tag{49}$$

and

$$T^m_{2p} = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}(m\pi)^{2r-2}(p+2r-1)}{(2r-1)!(p+r)(2p+2r-1)}. \tag{50}$$

These two equations can be readily derived from equations (18) and (23) by replacing sin $(m\pi x)$ and cos $(m\pi x)$, respectively, with their Taylor series expansions. Thus, one now essentially has two complementary sets of formulae which can practically cover any p values. It should be noted that S^m_{2p} and T^m_{2p} are simply some mathematical constants that are independent of the physical or geometrical variables of a plate.

In numerical calculations, equation (15) and hence equations (35–38) have to be truncated to the first, say, P terms.

Setting

$$Q_P = \sum_{p=0}^{P} \frac{(-1)^p k^{2p+1} \rho^{2p}}{(2p+1)!}, \tag{51}$$

the necessary condition for Q_P to converge to $(\sin k\rho)/\rho$ can be expressed as

$$P \geqslant \frac{ka\sqrt{1+\sigma^2}}{2} - 1. \tag{52}$$

Accordingly, one can write

$$\frac{\sin k\rho}{\rho} = Q_P + \delta_P, \tag{53}$$

where

$$|\delta_P| \leqslant \frac{k^{2P+3}(a^2+b^2)^{P+1}}{(2P+3)!} \quad \forall (x, y) \in [(0, a) \otimes (0, b)]. \tag{54}$$

Therefore, equation (54) can be used to determine how many terms have to be included to satisfy a pre-specified accuracy requirement.

Similarly, if only the first P terms are considered in equation (49) and

$$P \geqslant \frac{m\pi}{2}, \tag{55}$$

then its corresponding truncation error will be bounded by

$$|\delta_P| \leqslant \frac{k^{2P+3}(a^2+b^2)^{P+1}}{(2P+3)!}. \tag{56}$$

Obviously, this error estimate also applies to equation (50).

3. RESULTS AND DISCUSSIONS

3.1. CALCULATIONS OF THE EXPANSION COEFFICIENTS

As mentioned above equations (35–38) are theoretically good for arbitrary acoustic wavenumbers or frequencies. In calculations, however, one should be aware of the potential limits for the equations that are used to calculate the expansion coefficients, S^m_{2p} and T^m_{2p}, for various combinations of m and p values. It has been verified numerically that, regardless of m, equations (30) and (42) can be safely used for $p \leqslant 10$. Also, it was found that the index m is actually more suitable to be used to determine which set of formulae should be selected: while equations (30) and (42) are good for $m > 7$, equations (49) and (50) are better used for

$m \leqslant 7$. This confirms the earlier notion that these two sets of formulae complement each other very well. The values of S_{2p}^m and T_{2p}^m for m, $p \leqslant 10$ are listed in Tables 1 and 2.

3.2. RESULTS ON THE SELF- AND MUTUAL RADIATION RESISTANCE

For the sake of verifying the current formulae, the (self-) radiation resistances are shown in Figure 2 for the five lower order modes of a square plate. Since these curves have been repeatedly presented in the literature, there is no need for any further elaboration, other than simply pointing out the fact that they are the same as what is obtained from equation (7) or equation (9) through numerical integration. In Figure 3, the mutual radiation resistances are plotted for a few pairs of coupling modes. It is well known that the cross-modal coupling occurs only between a pair of modes having the same parity indices. For instance, the (1,1) mode can only couple with those modes that have odd indices in both x and y directions. It should be noted that all the curves in Figure 3 have been normalized by the self-radiation resistance for the (1,1) mode. The results show that the mutual radiation resistance for a pair of modes can be comparable to the self-resistances in a fairly wide frequency range which may well include the coincidence frequency of the higher mode. In addition, the degree of the coupling between any two (coupling) modes seems to decrease with their distance in the modal wavenumber space. More discussions about the characteristics of the cross-modal couplings can be found in reference [7].

3.3. EFFECTS OF THE PLATE ASPECT RATIO

Equations (35–38) clearly indicate that for a given wavenumber or frequency ka the self- and mutual radiation resistances are only a function of the plate aspect ratio. To demonstrate the effects of the aspect ratio, the self-radiation resistance for the (1,1) mode and the mutual radiation resistance resulting from its coupling with the (1,3) mode are, respectively, plotted in Figures 4 and 5 for the various aspect ratios: $\sigma = 1, 0{\cdot}8, 0{\cdot}6, 0{\cdot}4, 0{\cdot}2$. It is seen that, as the aspect ratio decreases, the humps on the self-radiation resistance curves become wider and shift toward high frequency, which contributes to the increase, by a factor of $\sqrt{1+\sigma^2}/\sigma$, of the modal wavenumber for a smaller aspect ratio. Again, the curves in Figure 5 have been divided by the self-radiation resistance for the (1,1) mode of the square plate. It is interesting to note that the (normalized) mutual radiation resistances for the two wider plates are maximal at $ka = 0$. However, as the plate shrinks in the y direction, the peak of the mutual radiation resistance curve moves to a much higher frequency. This raises a question on the popular belief that the effects of the cross-modal couplings are most remarkable at low frequencies.

For small ka the mutual radiation resistance is essentially proportional to the aspect ratio, manifested by the equal spacing in Figure 5 between any two adjacent curves near $ka = 0$. However, this is true only when both the modal indices are odd. If one of them is an even number, then the mutual radiation resistance, according to equations (46) and (48), will cubicly increase with the aspect ratio for small ka.

3.4. A COMPARISON OF THE CPU TIMES

In spite of its obvious academic value, one is still interested to know if the current solution has any computational benefits. To answer this question, the CPU time spent on calculating the self-radiation resistance of the (1,1) mode will be examined for a few different

Table 1

The expansion coefficients, S^m_{2p}

$2p$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 7$	$m = 8$	$m = 9$	$m = 10$
0	0·636620	0·000000	0·212207	0·000000	0·127324	0·000000	0·090946	0·000000	0·070736	0·000000
2	0·189304	− 0·159155	0·101325	− 0·079577	0·062630	− 0·053052	0·145097	− 0·039789	0·035191	− 0·031831
4	0·088144	− 0·110778	0·092415	− 0·073530	0·060616	− 0·051260	0·044354	− 0·039033	0·034840	− 0·031444
6	0·050384	− 0·074974	0·074891	− 0·065608	0·056292	− 0·048724	0·042721	− 0·037935	0·034060	− 0·030875
8	0·032433	− 0·052805	0·058089	− 0·056311	0·050886	− 0·045372	0·040526	− 0·036426	0·032982	− 0·030079
10	0·022561	− 0·038775	0·046437	− 0·047484	0·045101	− 0·041559	0·037931	− 0·034599	0·031655	− 0·029088
12	0·016574	− 0·029507	0·037096	− 0·039006	0·039534	− 0·037612	0·035120	− 0·032558	0·030141	− 0·027941
14	0·012679	− 0·023125	0·030095	− 0·033608	0·034501	− 0·033785	0·032256	− 0·030408	0·028506	− 0·026679
16	0·010006	− 0·018570	0·024789	− 0·028499	0·030103	− 0·030230	0·029465	− 0·028235	0·026810	− 0·025344
18	0·008094	− 0·015217	0·020707	− 0·024353	0·026329	− 0·027016	0·026829	− 0·026110	0·025106	− 0·023973
20	0·006680	− 0·012683	0·017518	− 0·020976	0·023114	− 0·024158	0·024392	− 0·024081	0·023434	− 0·022601

Table 2

The expansion coefficients, T^m_{2p}

$2p$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 7$	$m = 8$	$m = 9$	$m = 10$
0	0·405285	0·000000	0·045032	0·000000	0·016211	0·000000	0·008271	0·000000	0·005004	0·000000
2	0·038387	− 0·050661	0·020488	− 0·012665	0·007843	− 0·005629	0·004067	− 0·003166	0·002477	− 0·002026
4	0·009442	− 0·019863	0·016843	− 0·010740	0·007336	0·005249	0·003933	− 0·003046	0·002427	− 0·001977
6	0·003376	− 0·008771	0·010204	− 0·008401	0·006275	− 0·004727	0·003641	− 0·002874	0·002318	− 0·001905
8	0·001492	− 0·004366	0·006064	− 0·005983	0·005055	− 0·004069	0·003264	− 0·002644	0·002171	− 0·001807
10	0·000757	− 0·002308	0·003710	− 0·004148	0·003099	− 0·003379	0·002842	− 0·002377	0·001995	− 0·001687
12	0·000424	− 0·001406	0·002358	− 0·002881	0·002948	− 0·002736	0·002418	− 0·002094	0·001803	− 0·001553
14	0·000255	− 0·000078	0·001555	− 0·002028	0·002218	− 0·002184	0·002023	− 0·001816	0·001606	− 0·001412
16	0·000163	− 0·000574	0·001060	− 0·001453	0·001675	− 0·001733	0·001676	− 0·001557	0·001414	− 0·001270
18	0·000109	− 0·000391	0·000744	− 0·001060	0·001275	− 0·001374	0·001380	− 0·001324	0·001235	− 0·001132
20	0·000075	− 0·000275	0·000536	− 0·000787	0·000980	− 0·001093	0·001134	− 0·001120	0·001071	− 0·001003

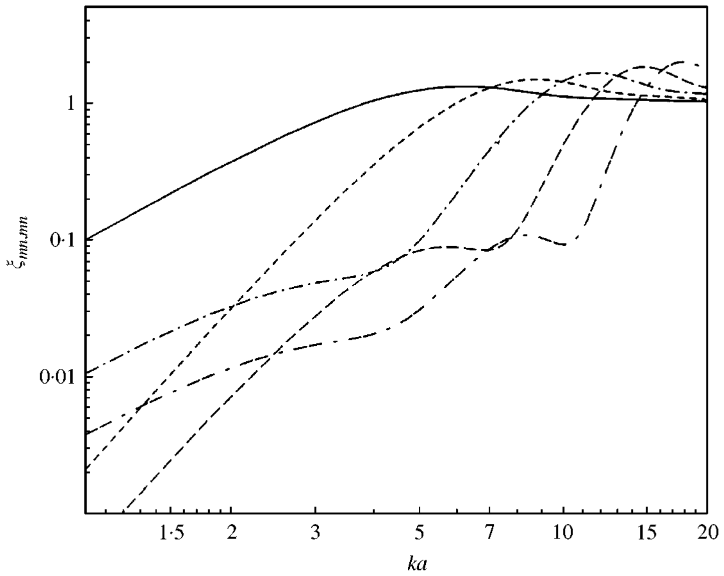

Figure 2. Self-radiation resistances for five lower order modes of a square plate: ——, (1,1); - - - - , (1,2); – · – ·, (1,3); – – – , (1,4); – – –, (1,5).

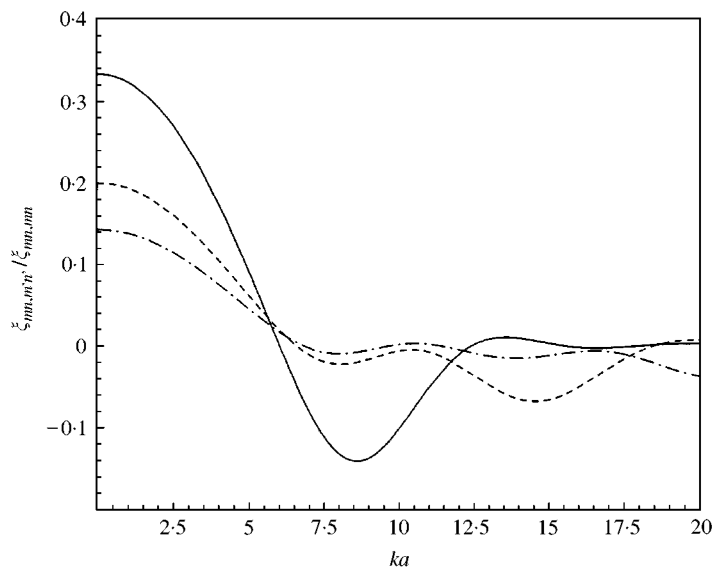

Figure 3. Mutual radiation resistances for a few pairs of coupling modes: ——, (1,1) × (1,3); - - - -, (1,1) × (1,5); – · – ·, (1,1) × (1,7).

frequencies. Since there is no other analytical or approximate solution readily available for an arbitrary frequency, the direct numerical integration of equation (9) will be used as a representative technique. To be statistically correct, at each frequency, equations (9) and (35) are repeatedly calculated 10 and 100 times respectively. The results shown in Table 3 are the averaged CPU times (in s) for each calculation. In Table 3, the numbers of terms used in equation (35) are also listed at each frequency, which are determined from

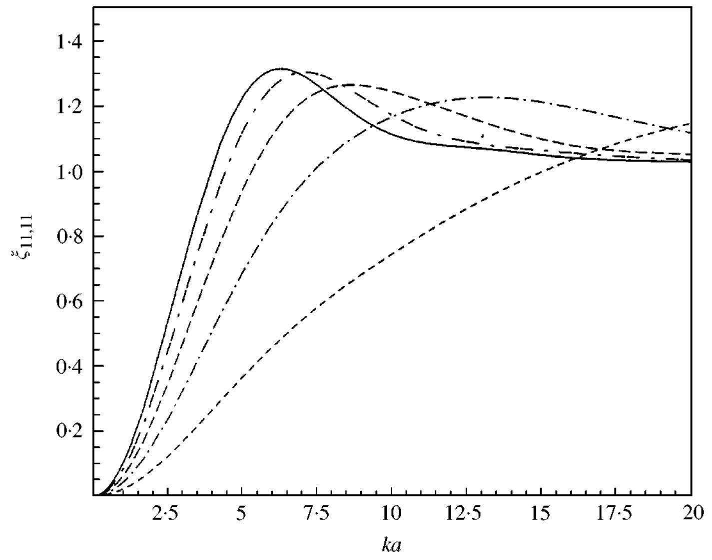

Figure 4. Influence of the aspect ratio on the self-radiation resistance for the (1,1) mode: - - - -, $\sigma = 0{\cdot}2$; – · – ·, $\sigma = 0{\cdot}4$; – – –, $\sigma = 0{\cdot}6$; – – -, $\sigma = 0{\cdot}8$; ——, $\sigma = 1$.

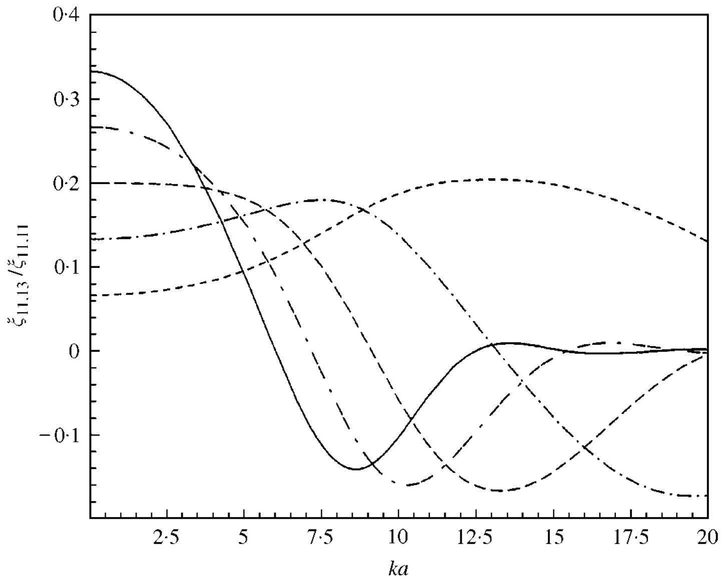

Figure 5. Influence of the aspect ratio on the mutual radiation resistance between the (1,1) and (1,3) modes: - - - -, $\sigma = 0{\cdot}2$; – · – ·, $\sigma = 0{\cdot}4$; – – –, $\sigma = 0{\cdot}6$; – – -, $\sigma = 0{\cdot}8$; ——, $\sigma = 1$.

equation (54) by setting $\delta_P = 10^{-8}$. It is seen that the current solution can reduce the CPU time by up to two orders of magnitude as compared with the numerical integration scheme.

Even though the above efficiency assessment is solely based on the CPU times spent on calculating the radiation resistance of the (1,1) mode, the conclusion is actually more true for other (higher order) modes. Because the number of terms used in the current solution is only a function of frequency, the amount of calculations involved is basically the same for all the modes, regardless of their modal indices. In contrast, however, the CPU time used in

TABLE 3

Comparison of the CPU times (s) spent on calculating the self-radiation resistance for the (1,1) mode

ka	Numerical integration, equation (9)	The present solution, equation (35)	Number of terms, P, used
1	1·175	0·0088	5
5	2·70	0·044	14
10	8·98	0·131	24
20	46·5	0·495	43

numerical integration can increase significantly with any of the modal indices. For instance, at $ka = 5$ the numerical integration uses 225 s for the (4,4) mode, compared to 2·7 s in Table 3 for the (1,1) mode. This should not come as a surprise by recognizing that at a given frequency the integrand corresponding to a higher order mode oscillates more rapidly over the integration domain, the plate surface. Taking account of this additional saving in the CPU time, one can expect that the current solution is far more efficient than what is already revealed in Table 3. It should be mentioned that all these calculations are performed on Mathematica [14].

4. CONCLUSIONS

A simple analytical solution in the form of series expansion has been developed for the self and mutual radiation resistances of a rectangular plate. Since no restriction is imposed upon the acoustic wavenumber, this solution is theoretically good for any acoustic wavenumbers or frequencies. In applications, it is particularly suitable for moderate acoustic wavenumbers that are often of primary concern in an acoustic analysis. In addition, given a (upper) frequency bound, the current formulae provide a simple and general way for deriving asymptotic or approximate solutions for the self and mutual radiation resistances.

Sufficient details are also given about the computational aspects of the current solution. Two complementary sets of formulae are derived for determining the expansion coefficients or constants that are independent of the acoustic wavenumber or the sizes of the plate. Numerical examples are presented to show the reliability of the current solution, and elucidate the reliance of the self and mutual radiation resistances on the acoustic wavenumber and on the plate aspect ratio. Finally, the current formulae are much more efficient than the traditional numerical integration technique in that the CPU time can be reduced by orders of magnitude.

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